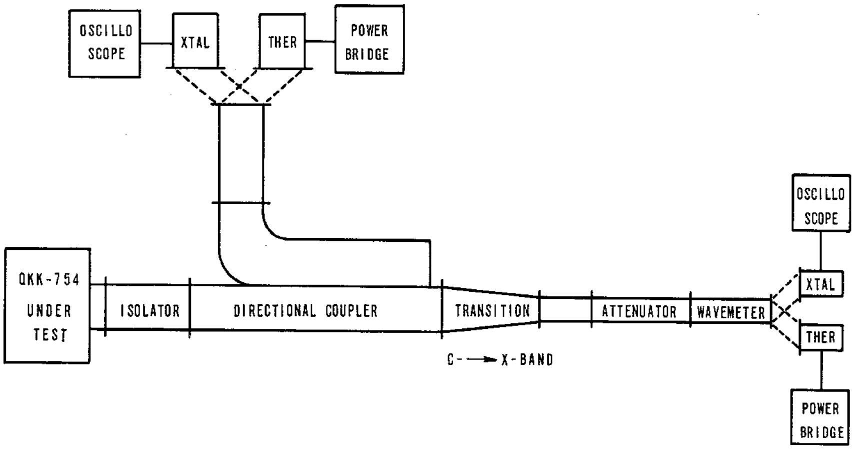

Fig. 1. Equipment used in the detection and measurement of *C*- and *X*-band power.

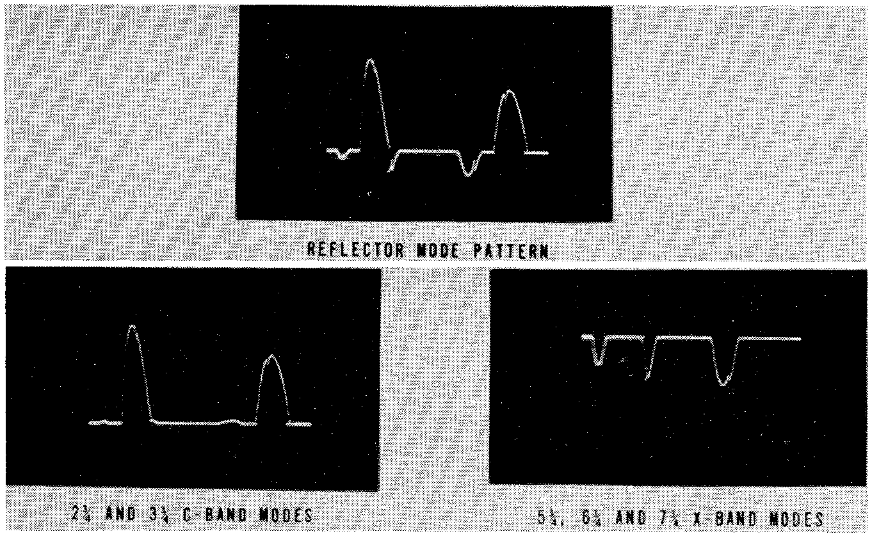

Fig. 2. Oscilloscope traces of *C*- and *X*-band reflector modes.

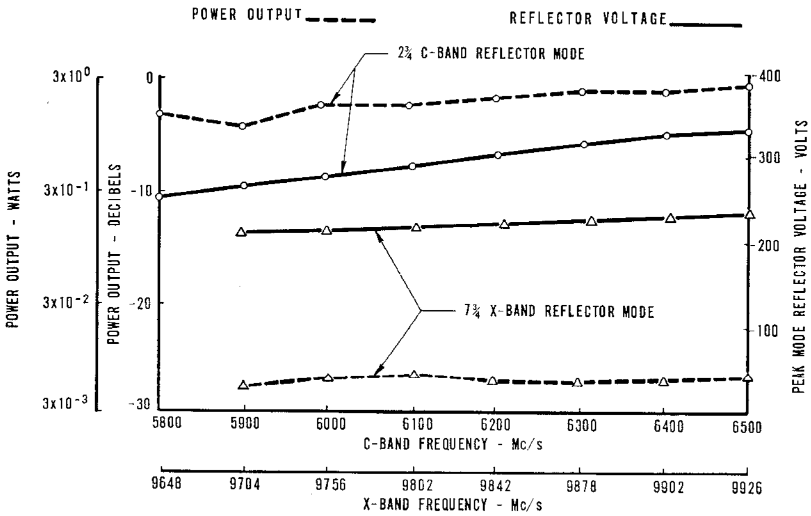

Fig. 3. Curves of power output and reflector voltage vs. frequency.

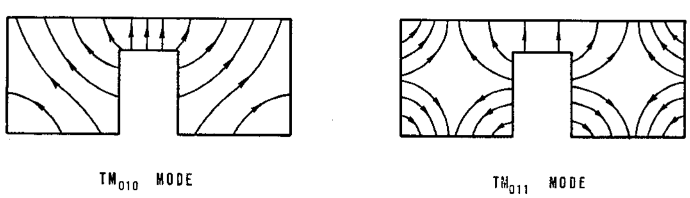

Fig. 4. Electric field configuration for beam interacting cavity modes.

ing figure of 9120 Mc/s fell within 7 percent of that of the detected output signal when the tube oscillated at a primary frequency of 6000 Mc/s.

It is worth noting that the amplitude of the secondary, higher frequency oscillation encountered in the above case may be amplified or suppressed through proper modification of the dimensional parameters of the resonator. The former has special significance for the millimeter region since it renders possible the operation of a reflex klystron at a frequency considerably higher than that which the physical dimensions of its circuit and beam would normally allow.

Acknowledgment

Thanks are due to R. W. Nersesian who performed most of the measurements reported in this correspondence.

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An Exact Analysis of Varactor Frequency Multipliers

A novel yet simple approach to the exact analysis of an abrupt-junction frequency doubler is presented, utilizing the fact that the voltage is proportional to the square of the charge. Penfield and Rafuse[1] were the first to consider the problem in an exact analysis. By imposing certain constraints they obtained useful design information with the aid of a large-scale computer. Through different constraints, the present analysis also offers an exact analysis, but the solution is expressed in a closed form. The series model of incremental elastance $S(t)$ and resistance R_s is shown in the circuit of Fig. 1.

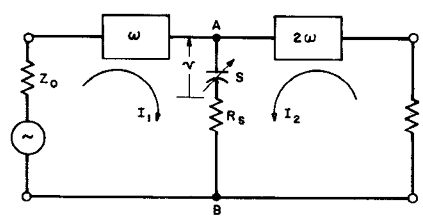

Fig. 1. Doubler circuit model.

The total current i and charge q flowing through the varactor diode are, respectively,

$$i = i_1 + i_2 = I_1 \cos \omega t + I_2 \cos (2\omega t + \theta) \quad (1)$$

and

$$q = \frac{I_1}{\omega} \sin \omega t + \frac{I_2}{2\omega} \sin (2\omega t + \theta) + K, \quad (2)$$

where θ is the phase angle between the fundamental and second harmonic in second harmonic time measure, and K is the average charge to be determined by the bound-

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[1] P. Penfield and R. P. Rafuse, *Varactor Applications*. Cambridge, Mass.: M.I.T. Press, 1962.

ary conditions such that

$$q_{\max} = Q_B, \quad \text{and} \quad q_{\min} = Q_\phi, \tag{3}$$

where Q_B and Q_ϕ are, respectively, the charge at the breakdown voltage V_B and contact potential ϕ. Throughout the analysis we assume the diode is fully driven, as implied by (3).

The voltage across the lossless abrupt-junction diode is

$$v = -\phi + \frac{S_{\max}^2}{4(V_B + \phi)} (q + Q_\phi)^2 \tag{4}$$

where $S_{\max}$ is the maximum elastance at V_B. Substituting (2) into (4), we obtain

$$v = V_b + V_1 \cos(\omega t + \zeta_1) + V_2 \cos(2\omega t + \zeta_2) + V_3 \cos(3\omega t + \theta) + V_4 \cos(4\omega t + 2\theta),$$

where

$$V_b = -\phi + M\left[2\omega^2(K + Q_\phi)^2 + \left(I_1^2 + \frac{I_2^2}{4}\right)\right], \quad \text{bias voltage} \tag{5}$$

$$V_1 = MI_1[16\omega^2(K + Q_\phi)^2 + I_2^2 - 8\omega I_2(K + Q_\phi)\sin\theta]^{1/2}, \tag{6}$$

$$V_2 = M[4\omega^2 I_2^2(K + Q_\phi)^2 + I_1^4 - 4\omega I_1^2 I_2(K + Q_\phi)\sin\theta]^{1/2}, \tag{7}$$

$$V_3 = -MI_1I_2, \qquad V_4 = -\tfrac{1}{4}MI_2^2, \tag{8}$$

$$M = \frac{S_{\max}^2}{8\omega^2(V_B + \phi)}, \tag{9}$$

$$\zeta_1 = \tan^{-1}\frac{I_2\sin\theta - 4\omega(K + Q_\phi)}{I_2\cos\theta}, \tag{10}$$

and

$$\zeta_2 = \tan^{-1}\frac{-2\omega I_2(K + Q_\phi)\cos\theta}{-I_1^2 + 2\omega I_2(K + Q_\phi)\sin\theta}. \tag{11}$$

Equations (6) to (8) are compatible with similar equations obtained by Penfield and Rafuse.[2] Note that the third and fourth harmonic voltages V_3 and V_4 also exist across the varactor terminals. Approximate analyses[3-8] published previously assume only the existence of the fundamental and second harmonic voltages V_1 and V_2 across the diode terminals and therefore cannot yield results in a self-consistent manner by constraining both currents and voltages.

Combining (6) and (10), and (7) and (11), we obtain the simple and *important relations:*

$$\cos\zeta_1 = \frac{1}{V_1} MI_1I_2\cos\theta \tag{12}$$

and

$$\cos\zeta_2 = \frac{1}{V_2} M[-I_1^2 + 2\omega I_2(K + Q_\phi)\sin\theta]. \tag{13}$$

The input power for the lossless case, or the input power to the "pure" nonlinear capacitance part of a diode only, is

$$P_1 = \tfrac{1}{2}V_1I_1\cos\zeta_1 = \tfrac{1}{2}MI_1^2I_2\cos\theta. \tag{14}$$

The power output from the lossless varactor is, similarly, according to the current convention used in Fig. 2:

$$P_2 = \tfrac{1}{2}V_2I_2\cos(\zeta_2 - \theta) = -\tfrac{1}{2}MI_1^2I_2\cos\theta = -P_1. \tag{15}$$

Depending upon the constraints imposed, the solutions of the nonlinear varactor problem are not unique. Penfield and Rafuse used maximum efficiency and maximum power transfer as the constraints. The present approach uses minimum dissipation and constant P_1 as constraints. The dissipation expression is, after substituting I_2 by (14),

$$P_d = \frac{1}{2}(I_1^2 + I_2^2)R_s = \frac{1}{2}\left[I_1^2 + \frac{4P_1^2}{M^2I_1^2\cos^2\theta}\right]R_s. \tag{16}$$

Setting $dP_d/dI_1 = 0$, we find the *optimization condition* as:

$$I_1 = \sqrt{2}\left(\frac{P_1}{M\cos\theta}\right)^{1/3} = \sqrt{2}I_2. \tag{17}$$

We next optimize the efficiency and the dissipation with respect to the phase angle θ. The efficiency, in general, is

$$\epsilon = \frac{\text{power output}}{\text{power input}} = \frac{P_1 - \frac{1}{2}I_2^2R_s}{P_1 + \frac{1}{2}I_1^2R_s}. \tag{18}$$

When the optimization condition (17) is used, (18) and (16) become, respectively,

$$\epsilon = \frac{2(P_1M^2\cos^2\theta)^{1/3} - R_s}{2(P_1M^2\cos^2\theta)^{1/3} + 2R_s}, \tag{19}$$

and

$$P_d = \frac{2}{3}R_s\left(\frac{P_1}{M\cos\theta}\right)^{2/3}. \tag{20}$$

Setting $d\epsilon/d\theta$ and $dP_d/d\theta$ equal to zero, we have proved that $\theta = 0°$ *is the condition for both maximum efficiency and minimum dissipation.* Equation (19) is plotted in Fig. 2 for the parameters shown.

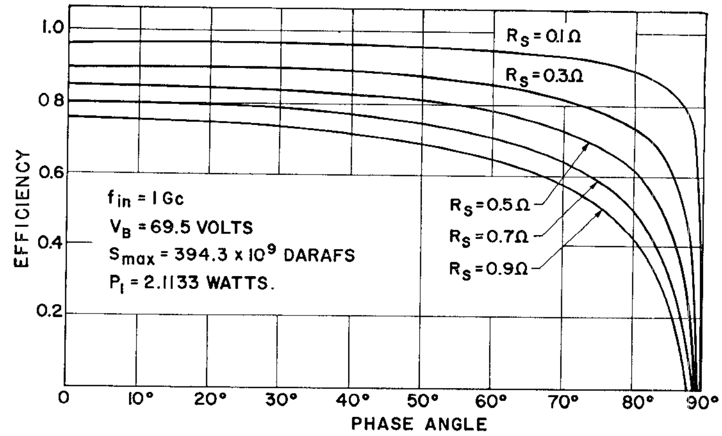

Fig. 2. The efficiency of a doubler as a function of phase angle.

The sought-after input impedance is

$$Z_{in} = \left(\frac{V_1}{I_1}\cos\zeta_1 + R_s\right) + j\left(\frac{V_1}{I_1}\sin\zeta_1\right), \tag{21}$$

and the sought-after load impedance is

$$Z_l = \left[\frac{V_2}{-I_2}\cos(\zeta_2 - \theta) - R_s\right] + j\left[\frac{V_2}{-I_2}\sin(\zeta_2 - \theta)\right]. \tag{22}$$

Taking the ratios of the resistive and reactive parts, respectively, from (21) and (22), we obtain the simple results:

$$\frac{R_{in}}{R_l} = \frac{1}{2\epsilon}; \quad \text{and} \quad \frac{X_{in}}{X_l} = -2 \quad (\text{for } \theta = 0°). \tag{23}$$

Accordingly, we discover the interesting fact that the input and load resistances are related through efficiency.

The remaining task of our problem is to find the current amplitude I_1 and the constant of integration K of (2). Using (2), setting $dq/d(\omega t) = 0$, and applying the boundary conditions (3) and the minimum dissipation constraint $I_1/I_2 = \sqrt{2}$ of (17), we obtain for the maximum efficiency case ($\theta = 0°$) that

$$I_1 = \frac{\omega(V_B + \phi)}{1.1775 S_{\max}}, \tag{24}$$

and

$$K = \frac{(V_B + \phi)}{S_{\max}}\left[1 - 2\left(\frac{\phi}{V_B + \phi}\right)^{1/2}\right]. \tag{25}$$

The efficiency, accordingly, becomes, by combining (18), (14), (17), and (24),

$$\epsilon = \frac{1 - 6.661\dfrac{\omega}{\omega_c}}{1 - 13.322\dfrac{\omega}{\omega_c}}, \tag{26}$$

where $\omega_c = S_{\max}/R_s$ is the "cutoff frequency." For low frequencies,

$$\epsilon \doteq 1 - 20\frac{\omega}{\omega_c}. \tag{27}$$

Figure 3 shows the comparison of (26) and Penfield-Rafuse results.

The usual given parameters for a doubler to be designed are the input frequency f, output power $P_{out} = \frac{1}{2}MI_1^2I_2 - \frac{1}{2}I_2^2R_s$, preferable breakdown voltage V_B, and intrinsic series resistance R_s. Combining (24), (9), and

[2] P. Penfield and R. P. Rafuse, *op. cit.*, p. 316.

[3] D. B. Leeson and S. Weinreb, "Frequency multiplication with nonlinear capacitors—a circuit analysis," *Proc. IRE*, vol. 47, pp. 2076–2084, December 1959.

[4] K. M. Johnson, "Large signal analysis of a parametric harmonic generator," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-8, pp. 525–532, September 1960.

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[7] D. B. Leeson, "Large signal analysis of parametric frequency multipliers," Solid-State Electronics Lab., Stanford University, Stanford, Calif., Tech. Rept. 1710-1, May 1962.

[8] K. K. N. Chang and P. E. Chase, "A rigorous analysis of harmonic generation using parametric diodes," *RCA Rev.*, vol. 24, pp. 214–225, June 1963.

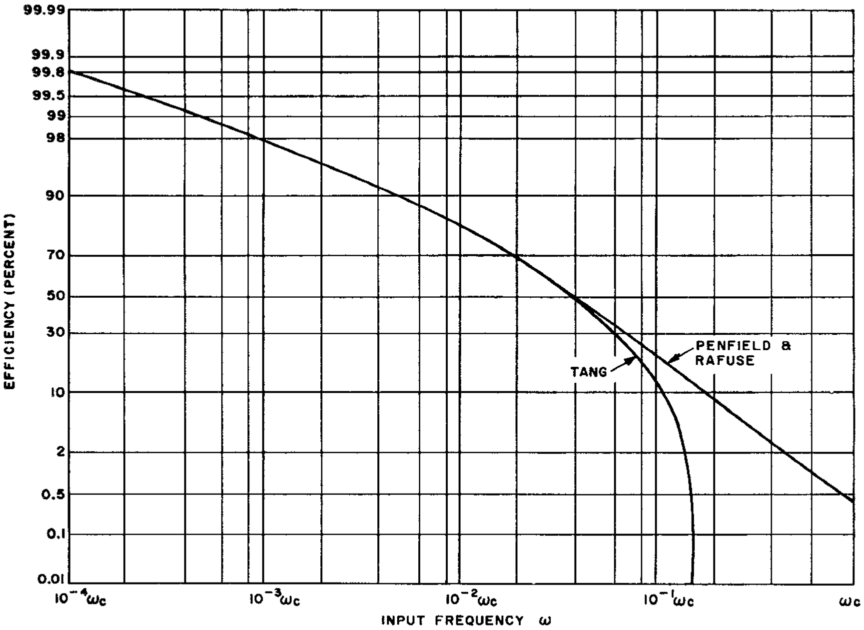

Fig. 3. The efficiency of a doubler as a function of ω.

P_{out} we obtain the value $S_{\max}$ of the sought-after varactor as:

$$S_{\max} = 0.08504 f(V_B + \phi)\left[(V_B + \phi) \pm \sqrt{(V_B + \phi)^2 - 984.27 R_s P_{\text{out}}}\right]/P_{\text{out}}. \quad (28)$$

The other sought-after quantities are:

$$R_l = 0.02389\left(\frac{S_{\max}}{f}\right) - R_s \quad (29)$$

$$R_{\text{in}} = 0.011945\left(\frac{S_{\max}}{f}\right) + R_s = \frac{R_l}{2\epsilon} \quad (30)$$

$$L_l(\text{load inductance}) = 0.0031662\,\frac{S_{\max}}{f^2} \quad (31)$$

$$L_0(\text{source impedance required}) = 4L_l \quad (32)$$

$$V_b(\text{bias voltage}) = 0.3596(V_B + \phi) - \phi, \quad (33)$$

and

$$P_d = 19.9327 R_s f^2 (V_B + \phi)^2 / S_{\max}^2. \quad (34)$$

In passing, we shall supply the simple proof that except for the doubler, any abrupt-junction diode frequency multiplier without an idler is not possible. Let the current and charge flowing through the diode be

$$i = i_1 + i_n = I_1 \cos \omega t + I_n \cos (n\omega t + \theta), \quad (35)$$

and

$$\begin{aligned} q &= q_1 + q_n \\ &= Q_1 \sin \omega t + Q_n \sin (n\omega t + \theta) + K. \quad (36) \end{aligned}$$

Using (4), we have the voltage across the diode

$$\begin{aligned} v = -\phi + \frac{S_{\max}^2}{4(V_B + \phi)} &\cdot \left\{\left[\frac{1}{2}(Q_1^2 + Q_n^2) + (K + Q_\phi)^2\right]\right. \\ &+ 2Q_1(K + Q_\phi) \sin \omega t - \frac{Q_1^2}{2} \cos 2\omega t \\ &+ Q_1 Q_n \cos [(n-1)\omega t + \theta] \\ &+ 2Q_n(K + Q_\phi) \sin (n\omega t + \theta) \\ &- Q_1 Q_n \cos [(n+1)\omega t + \theta] \\ &\left. - \frac{Q_n^2}{2} \cos (2n\omega t + 2\theta)\right\}. \quad (37) \end{aligned}$$

Comparing (35) with (37), we notice that except for $n=2$ case the fundamental voltage is always in quadrature with the fundamental current and, likewise, nth harmonic voltage is always in quadrature with nth harmonic current.

On the other hand, the same approach can be used for frequency multipliers other than the abrupt-junction doubler by adding the necessary idler current or currents flowing through the varactor as constraints.

It can be shown that doublers are possible for hyperabrupt junctions with γ(doping profile exponent) $= \frac{2}{3}$ and quadruplers without idler are possible for the hyperabrupt functions with $\gamma = \frac{3}{4}$.

The results obtained by the present approach are quite different from those of the Penfield-Rafuse approach using Fourier expansion of nonlinear elastance. The present approach emphasizing minimum dissipation can be particularly useful in cases where dissipation is the principle problem; for example, for high power varactor multipliers, minimum dissipation is often the design objective, not maximum efficiency or maximum power output.

The present approach is simple and straightforward in concept and does not require the aid of a computer.

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Optimum Pitch of Traveling-Wave Masers

The purpose of this correspondence is to show theoretically that there exists an optimum pitch (p in Fig. 1), which gives maximum net gain for traveling-wave masers (TWM) [1], utilizing transverse strip slow wave structures (e.g., comb-structures [2], Karp-structures [3], and meander lines). The net gain in dB of the traveling-wave maser may be expressed [1] as

$$G = 27\,\frac{L}{\lambda_0}\,s\left[\frac{1}{|Q_m|} - \frac{1}{Q_0} - \frac{1}{Q_i}\right] \quad (1)$$

where L is the structure length, λ_0 the free space wavelength, s the slowing factor, $|Q_m|$ the magnetic quality factor of the maser material, Q_0 the ohmic Q-factor, and Q_i the Q-factor related to the forward wave losses in the isolator. The Q-factors depend on the particular structure geometry.

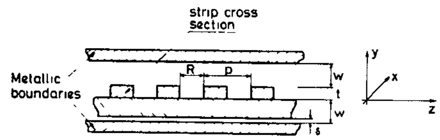

Fig. 1. A typical cross section of transverse strip slow-wave structures.

A typical cross section of a transverse strip slow-wave structure is shown in Fig. 1. If the cross-sectional dimensions within one period p are much smaller than the free space wavelength, and if the strip lengths are long compared to their cross sections, the fields may be approximated by TEM waves traveling along the strips (the x direction of Fig. 1).

Assume the pitch of a particular slow-wave structure is p_0. The slowing of the structure becomes s_0 and the ohmic Q-factor Q_{00}. A scaling of the cross section is obtained when all cross-sectional dimensions are multiplied by the factor p/p_0. Hence, the new pitch becomes p.

The impedance of the TEM waves $K(\phi)$ is only dependent on the relative cross-sectional dimensions and, consequently, remains constant during the scaling procedure. $K(\phi)$ does, however, depend on the phase shift ϕ between two strips [2], [3]. If the propagation constant of the wave traveling along the structure (the z direction of Fig. 1) is β, we have

$$\phi = \beta \cdot p.$$

The $\omega - \phi$ characteristic of the structure is determined by the boundary conditions of the TEM waves, the length of the strips, and the impedance $K(\phi)$ [2], [3]. We now assume that the boundary conditions, the length of the strips, and the cross-sectional dimensions divided by p is constant during a scaling of the cross section by p/p_0. Hence, the $\omega - \phi$ characteristic is independent of p. The group velocity v_g and the slowing s of the wave traveling in the z direction becomes

$$v_g = \frac{d\omega}{d\beta} = p\,\frac{d\omega}{d\phi}$$

$$s = \frac{c_0}{v_g} = \frac{c_0}{p_0 \cdot \dfrac{d\omega}{d\phi}}\,\frac{p_0}{p} = s_0 \cdot \frac{p_0}{p} \quad (2)$$

where c_0 is the velocity of light in vacuum.

The ohmic quality factor Q_0 is proportional to V/S for constant relative cross-sectional dimensions, where V is the volume and S is the surface of one period of the struc-

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