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Linear Superposition: Tutorial on Thermal-RC Networks

Transient Analysis in Linear Thermal Systems

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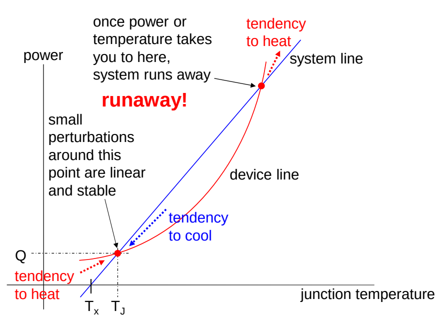
Outline

- Linear Systems fundamentals
  - why
  - what
  - some useful consequences
    - linear superposition
    - reciprocity
- Thermal RC Networks
- Case study – Calibration Chamber
  - design
  - step response experiments
  - fitting to measured data
  - predicting control input
  - testing predictions

Linear Systems - why

- Why?
  - Because they can be analyzed!
    - frequency domain, modal analysis
    - time domain, s-plane, Laplace Transforms
  - Many real systems are sufficiently linear (meaning, over a useful range) that it's worth the effort
- Danger
  - If you forget, in the end, that you *assumed* linearity, you can go way wrong
    - thermal runaway provides a specific example

Thermal Runaway



Linear Systems - why

- Why?
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    - time domain, s-plane, Laplace Transforms
  - Many real systems are sufficiently linear (meaning, over a useful range) that it's worth the effort
- Danger
  - If you forget, in the end, that you *assumed* linearity, you can go way wrong
    - thermal runaway provides a specific example
  - always prudent to review the range of the results to be sure you haven't gone too far outside the linear window, and what the implications might be

Linear Systems – what

- Changes in the output variables are *proportional* to changes in the input variables

Outputs Proportional to Inputs

$$\Delta T_1 = a_{11} \cdot q_1 + a_{12} \cdot q_2 + \cdots + a_{1n} \cdot q_n$$

$$\Delta T_2 = a_{21} \cdot q_1 + a_{22} \cdot q_2 + \cdots + a_{2n} \cdot q_n$$

$$\vdots$$

$$\Delta T_m = a_{m1} \cdot q_1 + a_{m2} \cdot q_2 + \cdots + a_{mn} \cdot q_n$$

Linear Systems – what

- Changes in the output variables are *proportional* to changes in the input variables
- When time is considered, governing differential equations are linear

Governing Differential Equations are Linear

$$k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} + k_z \frac{\partial^2 T}{\partial z^2} = \rho c_p \frac{\partial T}{\partial t}$$

$$q = hA \cdot (T - T_A)$$

$$C_{th} \frac{dT}{dt} = q + \frac{T - T_A}{R_{th}}$$

Linear Systems – what

- Changes in the output variables are *proportional* to changes in the input variables
- When time is considered, governing differential equations are linear
- In the thermal domain
  - these are *constant*:
    - thermal conductivity
    - heat capacity
    - density
    - convection film coefficients

Coefficients are *constant*

$$k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} + k_z \frac{\partial^2 T}{\partial z^2} = \rho c_p \frac{\partial T}{\partial t}$$

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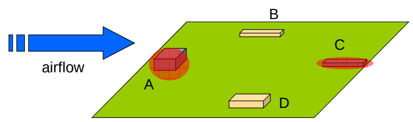
  - these restrictions generally apply:
    - convection is in algebraic proportion to temperature differences

Non-symmetric convection

- Upwind and downwind in forced-convection dominated applications



Heat in at "A" will raise temperature of "C" more than heat in at "C" will raise temperature of "A"

Linear Systems – what

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  - these are *constant*:
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    - heat capacity
    - density
    - convection film coefficients
  - these restrictions generally apply:
    - convection is in algebraic proportion to temperature differences
    - radiation and free convection are inherently non-linear

Radiation Resistance

basic heat transfer relationship for surface radiation is 4th-order in temperature, so

$$R = \frac{1}{\sigma \varepsilon F A \left(T^2 + T_a^2\right)\left(T + T_a\right)}$$

obviously not really a *constant*

Free Convection

$$R = \frac{1}{hA}$$

not explicitly temperature dependent, but ***h***, the film coefficient, depends on density, which is highly temperature dependent

Poll Question #1

- Which of the following are intrinsically non-linear …
  a) Conduction heat transfer
  b) Free convection heat transfer
  c) Forced convection heat transfer
  d) Radiation heat transfer

(choose all that apply)

Linear System Consequences

- If outputs are proportional to inputs, then the *inverse* problem (exchanging roles of inputs and outputs) is also linear

Role reversal

- Temperature can be the output, and power the input

$$T_j = \theta_{Jx} \cdot q + T_x$$

- or, power can be the result, and temperature the input

$$q = \frac{1}{\theta_{Jx}} \cdot T_j - \frac{T_x}{\theta_{Jx}}$$

- either way, it's a linear equation

Linear Algebra Matrix Equivalent

- Temperature can be the output, and power the input

$$\overline{T}_j = [\theta_{Jx}] \cdot \overline{q} + \overline{T}_x$$

- or, power can be the result, and temperature the input

$$\overline{q} = [\theta_{Jx}]^{-1} \cdot \overline{T}_j - [\theta_{Jx}]^{-1} \cdot \overline{T}_x$$

- either way, it's a linear equation

Linear System Consequences

- If outputs are proportional to inputs, then the *inverse* problem (exchanging roles of inputs and outputs) is also linear
  - In thermal systems, you can think of temperature as an output and power (heat) as the input, or you can think of temperature as the input and heat flow (power) as the output; either way, the problem is linear
- Linear superposition
  - solutions to particular cases can be linearly combined to create solutions to other cases

Linear superposition – what is it?

- The total response of a point within the system, to excitations at all points of the system, is the sum of the individual responses to each excitation taken independently.

$$\Delta T_{\text{composite}} = \Delta T_{\text{source 1}} + \Delta T_{\text{source 2}} + \cdots + \Delta T_{\text{source n}}$$

Linear System Consequences

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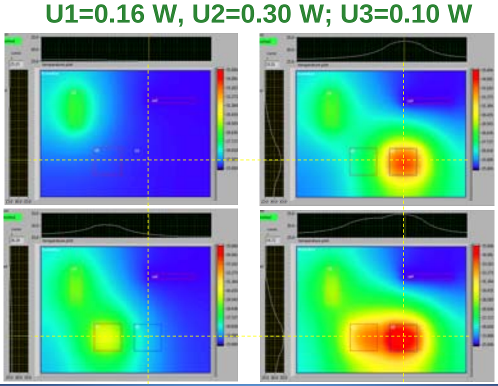
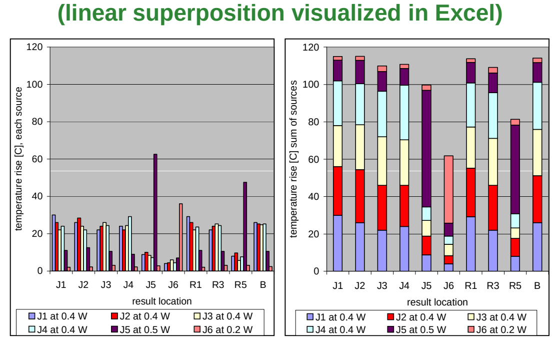
  - applies both spatially and temporally

U1=0.16 W, U2=0.30 W; U3=0.10 W



Bar Charts and Stacked Bar Charts (linear superposition visualized in Excel)



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  - solutions to particular cases can be linearly combined to create solutions to other cases
  - applies both spatially and temporally

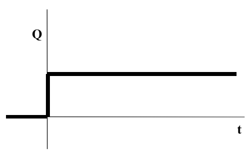
Linear Superposition - temporally

- The combined response of a point to multiple independent time-varying inputs is the sum of the transient response of each individual time-varying input

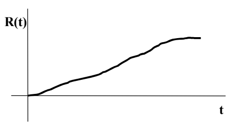
$$\boxed{\Delta T\left(f_1(t), f_2(t)\ldots f_n(t)\right)} = \boxed{\Delta T\left(f_1(t)\right)} + \boxed{\Delta T\left(f_2(t)\right)} + \cdots + \boxed{\Delta T\left(f_n(t)\right)}$$

Basic Heating Curve - a “Single Pulse”

step input of power

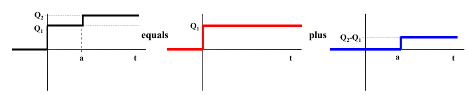


transient response curve

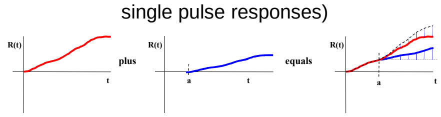


Double Step

double step, decomposed into two single steps

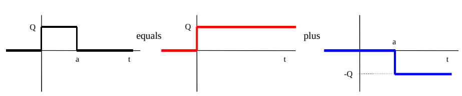


Temperature response of double step (constructed from superposition of two single pulse responses)

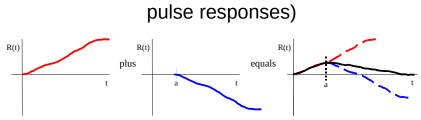


Single Square Pulse

Finite pulse, decomposed into two infinite steps

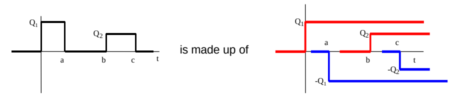


Temperature response of a finite pulse (constructed from superposition of two single pulse responses)

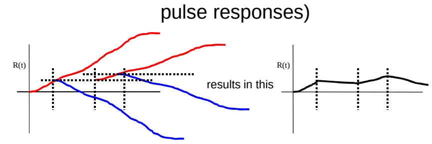


Two Differing Pulses

Two finite pulses decomposed into infinite steps

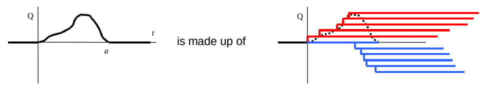


Temperature response for two finite pulses (constructed from superposition of four single pulse responses)

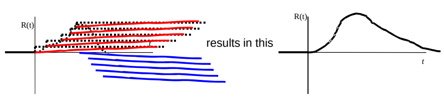


An Arbitrary Pulse

An arbitrary pulse decomposed into several infinite steps



Temperature response constructed for this arbitrary pulse

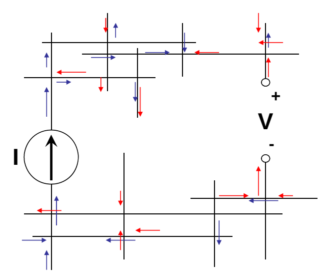


Linear System Consequences

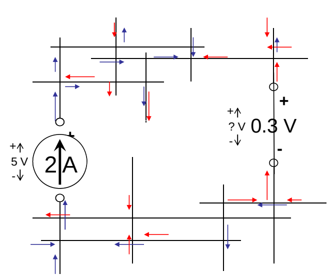
- If outputs are proportional to inputs, then the *inverse* problem (exchanging roles of inputs and outputs) is also linear
  - In thermal systems, you can think of temperature as an output and power (heat) as the input, or you can think of temperature as the input and heat flow (power) as the output; either way, the problem is linear
- Linear superposition
  - solutions to particular cases can be linearly combined to create solutions to other cases
  - applies both spatially and temporally
- Reciprocity
  - a particular sort of mathematical symmetry that arises in fully linear systems

The Reciprocity Theorem

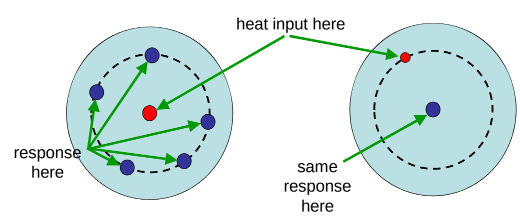
Electrical Reciprocity



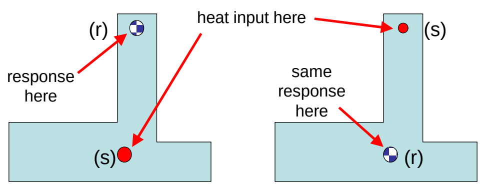
Electrical Reciprocity



Thermal Reciprocity

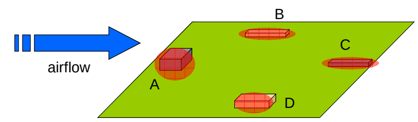


Another Thermal Reciprocity Example



When Does Reciprocity *NOT* Apply?

- Upwind and downwind in forced-convection dominated applications



Heat in at "A" will raise temperature of "C" more than heat in at "C" will raise temperature of "A"

"B" and "D" may still be roughly reciprocal

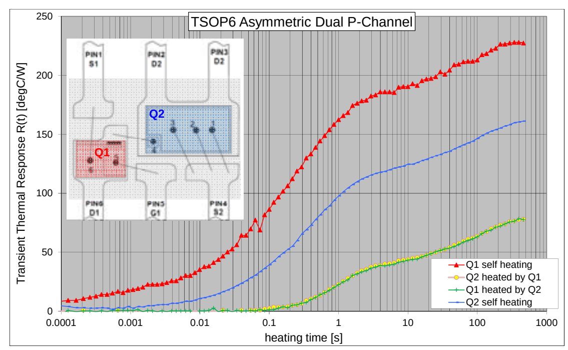
(Square part of) Matrix is Symmetric

columns are the "n" heat sources

rows are the "m" response locations

J1	75	65	55	60	22	10
J2	65	71	60	55	25	11
J3	55	60	65	61	21	15
J4	60	55	61	73	18	11
J5	22	25	21	18	125	14
J6	10	11	15	11	14	180
R1	73	65	55	59	22	10
R3	55	60	63	61	21	15
R5	20	24	14	19	95	15
B	65	63	62	63	21	12

Transient Reciprocity



Poll Question #2

- Reciprocity …
  - a) Relates temperature inputs and temperature responses
  - b) Is a covert CIA program to take out drug cartels
  - c) Relates heat inputs and temperature responses
  - d) Never applies to linear systems

(choose all that apply)

Outline

- Linear Systems fundamentals
  - why
  - what
  - some useful consequences
    - linear superposition
      - in space and time
    - reciprocity
- **Thermal RC Networks**
- Case study – Calibration Chamber
  - design
  - step response experiments
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Thermal RC Networks

- Thermal/Electrical analogy
- Grounded vs. Non-grounded
  - physical significance
  - mathematical convenience
  - interchangeability
- 1-rung model(s)
- 2-rung models
  - "tau" is not RC product except when …

Thermal / Electrical Analogy

temperature	<=>	voltage
power	<=>	current
temp/power	<=>	resistance
energy/degree	<=>	capacitance

Conduction Resistance

basic heat transfer relationship for 1-D conduction

$$q = k \cdot A \cdot \frac{dT}{dx} \quad \approx \quad k \cdot A \cdot \frac{\Delta T}{L}$$

if we define

$$R = \frac{\Delta T}{q}$$

then

$$R = \frac{L}{k \cdot A}$$

Convection Resistance

basic heat transfer relationship for surface cooling

$$q = h \cdot A \cdot \Delta T$$

if we define

$$R = \frac{\Delta T}{q}$$

then

$$R = \frac{1}{hA}$$

Radiation resistance

basic heat transfer relationship for surface radiation

$$\begin{aligned} q &= \sigma \cdot \varepsilon \cdot F \cdot A \cdot \left(T^4 - T_a^4\right) \\ &= \sigma\varepsilon F A\left(T^2 + T_a^2\right) \cdot \left(T^2 - T_a^2\right) \\ &= \sigma\varepsilon F A\left(T^2 + T_a^2\right)\left(T + T_a\right) \cdot \left(T - T_a\right) \\ &= \sigma\varepsilon F A\left(T^2 + T_a^2\right)\left(T + T_a\right) \cdot \Delta T \end{aligned}$$

if we define

$$R = \frac{\Delta T}{q}$$

then

$$R = \frac{1}{\sigma\varepsilon F A\left(T^2 + T_a^2\right)\left(T + T_a\right)}$$

obviously not really a *constant*

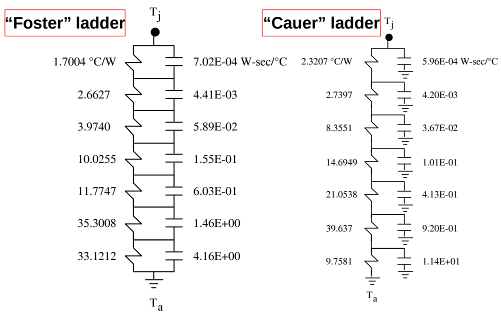
Models: Grounded vs. Non-Grounded

("Cauer" ladders) ("Foster" ladders)

- Physical significance: if thermal capacitors are grounded, they bear some relationship to a physical system; *not so for non-grounded C's*
- Mathematical convenience: certain non-grounded networks (ladders) are mathematically trivial
- Interchangeability: single-input thermal systems can be represented as either grounded or non-grounded

Actual Thermal RC Networks

Non-Grounded vs. Grounded C comparison

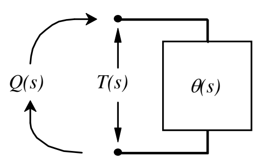


Laplace Transform Basics

item	time domain	s-plane	frequency domain
function	$f(t)$	$F(s)=\int_0^\infty e^{-st} f(t)dt$	
derivative	$\frac{df}{dt}$	$sF(s) \quad [-f(0)]$	
resistor	$v = Ri$	$Z_R = R$	$Z_R = R$
capacitor	$i = C\frac{dv}{dt}$	$Z_C = \frac{1}{sC}$	$Z_C = \frac{1}{j\omega C}$
inductor	$v = L\frac{di}{dt}$	$Z_L = sL$	$Z_L = j\omega L$
exponential	e^{-at}	$\frac{1}{s+a}$	

Laplace Transform Basics, cont'

- Your basic thermal network:



by definition,

$$\theta(s) = \frac{T(s)}{Q(s)}$$

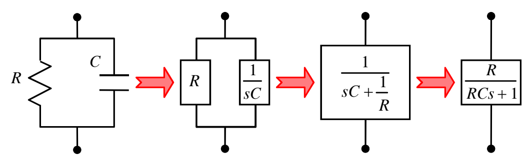
so

$$T(s) = \theta(s) \cdot Q(s)$$

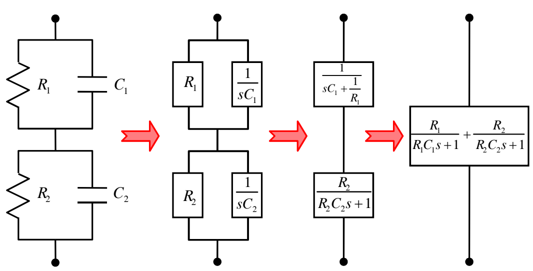
or

$$Q(s) = \theta^{-1}(s) \cdot T(s)$$

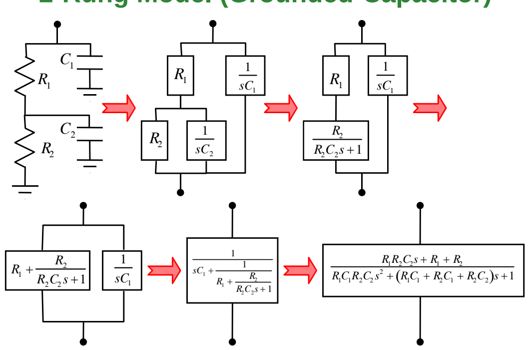
1-Rung Model(s)



2-Rung Model (Non-Grounded Capacitor)



2-Rung Model (Grounded-Capacitor)



Compare the Transfer Functions

From the non-grounded-capacitor model

$$\frac{R_1}{R_1C_1s+1}+\frac{R_2}{R_2C_2s+1}=\frac{(C_1+C_2)R_1R_2s+(R_1+R_2)}{R_1C_1R_2C_2s^2+(R_1C_1+R_2C_2)s+1}$$

From the grounded-capacitor model:

$$\frac{R_1R_2C_2s+(R_1+R_2)}{R_1C_1R_2C_2s^2+(R_1C_1+R_2C_1+R_2C_2)s+1}$$

2-Rung Models Compared

- Time constants are roots of denominators
- "tau" is not RC product when capacitors are grounded!

non-grounded capacitors: (Foster)

$$\frac{R_1}{R_1C_1s+1}+\frac{R_2}{R_2C_2s+1}$$

can be written

$$\frac{1}{C_1}\frac{1}{s+\frac{1}{\tau_1}}+\frac{1}{C_2}\frac{1}{s+\frac{1}{\tau_2}}$$

where time constants are

$\tau_1=R_1C_1 \quad \tau_2=R_2C_2$

grounded-capacitors: (Cauer)

$$\frac{R_1R_2C_2s+R_1+R_2}{R_1C_1R_2C_2s^2+(R_1C_1+R_2C_1+R_2C_2)s+1}$$

can be written

$$\frac{1}{C_1}\frac{s+\frac{R_1+R_2}{R_1R_2C_2}}{\left(s+\frac{1}{\tau_1}\right)\left(s+\frac{1}{\tau_2}\right)}$$

where time constants are

$$\tau_1=\frac{2\cdot R_1C_1}{\left(1+\frac{c}{r}+c\right)+\sqrt{\left(1+\frac{c}{r}+c\right)^2-4\frac{c}{r}}}$$

$$\tau_2=\frac{2\cdot R_2C_2}{\left(1+\frac{r}{c}+r\right)-\sqrt{\left(1+\frac{r}{c}+r\right)^2-4\frac{r}{c}}}$$

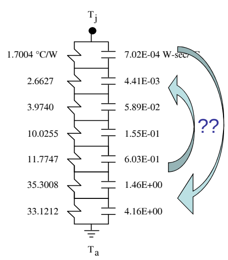
$$\left(\text{and defining} \quad r=\frac{R_2}{R_1} \quad c=\frac{C_1}{C_2}\right)$$

r	c	$\frac{\tau_1}{R_1C_1}$	$\frac{\tau_2}{R_2C_2}$
100	0.01	0.99	1.01
10	0.1	0.91	1.10
3	1/3	0.73	1.36
1	0.1	0.90	1.11
1	1	0.38	2.62

Interesting (and *Important*) Implications

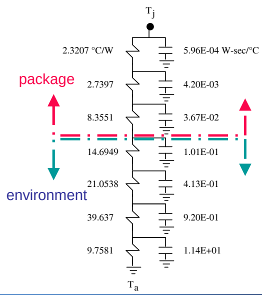
"Foster" ladder

Rungs can be in any order and T_j has identical behavior! So where do you "split" it?



"Cauer" ladder

Order matters, so a "split" *can* make good physical sense.



RC Networks in Excel®

Foster ladders of any number of rungs, having simple RC products and associated closed-form transient response, can be easily built and computed using Excel techniques.

A Foster ladder even can be converted to a Cauer equivalent using Excel techniques. (However, there is no a *priori* guarantee that all R's and C's will be positive!)

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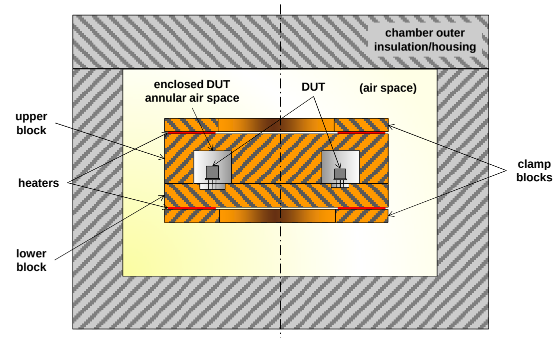
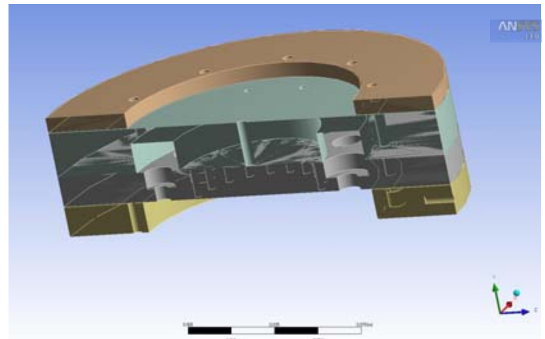
Temperature Chamber Design Goals

- Test 5 DUT's simultaneously
  - package types: TO18 and TO39
- Range: room temperature to 200 °C
- Hold +/- 0.1 °C uniformity across all DUT's
- Make a 45 °C temperature step with about 0.1 °C accuracy, and be stable within about 15 minutes

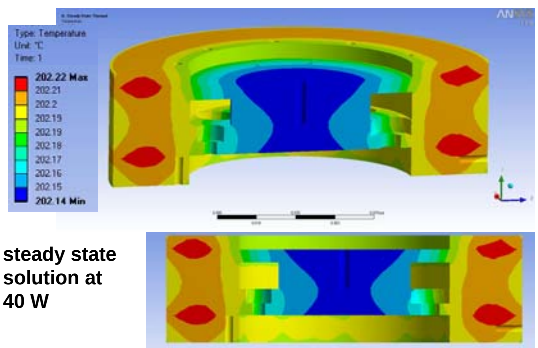
Temperature Chamber Schematic



FEM of Inner Heater Block Assembly



FEM Steady State Results

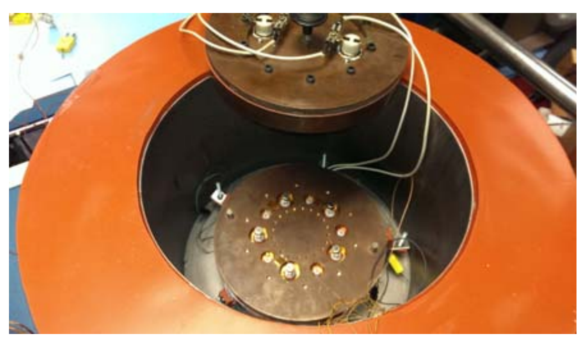


steady state solution at 40 W

Looking into chamber, lid removed

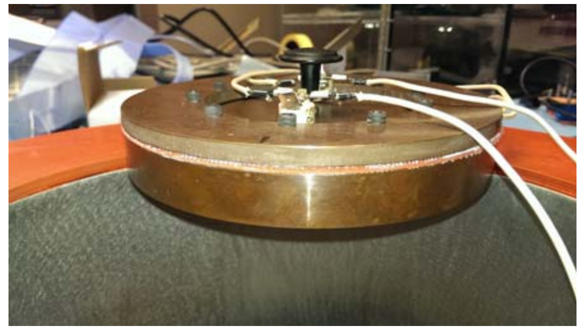
Closer view of isolating air space

Upper heater block lifted off

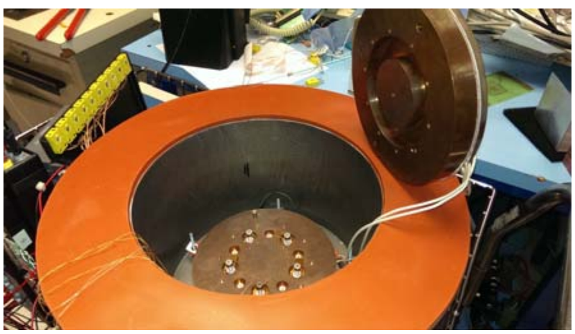


View of DUT cavities in lower heater block

Side view of upper block and heater clamp



Another view of opened heater assembly



Lid ready to be placed onto chamber

Chamber, temperature controller & scanner

Poll Question #3

- Roger's high-temperature calibration chamber most resembles:
  a) A neutrino detector
  b) A stainless steel hatbox
  c) A bomb calorimeter
  d) A phlogiston ambivalizer

(choose all that apply)

Back-of-the-Envelope Estimates

Name of Part	Volume*	dens	cp	Mth	Cth
	m^3	kg/m^3	J/kg/°C	kg	J/°C
Lower Block	2.72E-4	8900	383	2.42	927
Upper Block	3.73E-4	8900	383	3.32	1272
Upper Outer Ring	9.43E-5	8900	383	0.84	321
Lower Outer Ring	1.86E-4	8900	383	1.65	633
Bottom Cover Plate	3.45E-5	8900	383	0.31	118
total	9.59E-4		total	8.54	3270

inside of chamber				
geometry				
dia	10	inch	0.254	m
height	8	inch	0.2032	m
wall thickness	0.125	inch	0.003175	m
lid/wall overlap	3	inch	0.0762	m
surface area				
floor			0.0507	m^2
ceiling			0.0507	m^2
wall main			0.1621	m^2
wall overlap			0.0608	m^2
		total	0.3851	m^2
		volume	0.00122	m^3
		steel dens	7850	kg/m^3
		steel cp	460	J/kg/°C
		mass	9.60	kg
			4415	J/°C
convection resistance of inside chamber walls				
		film coefficient	13	W/m^2/°C
		R-conv	0.20	°C/W

convection resistance of inner assembly to air space				
		surface area	0.0669	m^2
		film coefficient	13	W/m^2/°C
		R-conv	1.2	°C/W
outside of chamber				
geometry				
O.D.	16	inch	0.4064	m
I.D.	10	inch	0.254	m
height	16	inch	0.4064	m
wall thickness	0.125	inch	0.003175	m
surface area				
top			0.1297	m^2
bottom			0.1297	m^2
sides			0.5189	m^2
cover/base overlap			0.0790	m^2
		total	0.9364	m^2
		volume	0.00297	m^3
		steel dens	7850	kg/m^3
		steel cp	460	J/kg/°C
		mass	23.34	kg
			10736	J/°C
total surface area of outside chamber				
			0.7783027	m^2
		film coefficient	3.2	W/m^2/°C
		R-conv	0.40	°C/W
conduction resistance of rock wool insulation				
		thickness	0.0762	m
		area	0.521	m^2
		conductivity	0.055	W/m/°C
		R	2.7	°C/W

More Back-of-the-Envelope (transient) Calculations

- Minimum ramp time

$$\Delta E = C_{th}\Delta T$$

$$time = \frac{\Delta E}{power}$$

step	45	°C
power	230	W
ramp energy	147142	J
ramp time	640	s

- Thermal diffusion time

$$time = \frac{L^2}{\alpha}$$

Cu diffusivity	8.99E-3	s/mm^2		
upper clamp	3	mm	0.08	s
lower clamp	6	mm	0.32	s
block thick	25	mm	5.6	s
block radius	75	mm	51	s

Even More Back-of-the-Envelope Calculations

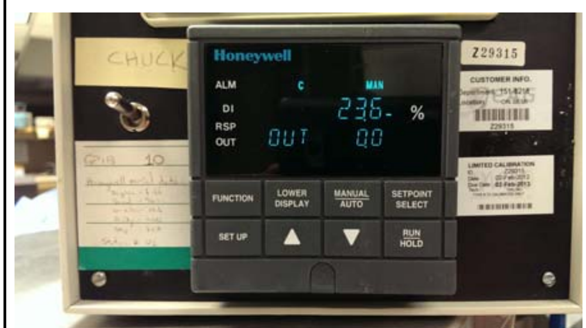
- RC time constant of heater block assembly

$$\begin{aligned}\tau &= R_{conv}C_{th}\\ &= 1.4\cdot 3300\\ &= 4600\ s\end{aligned}$$

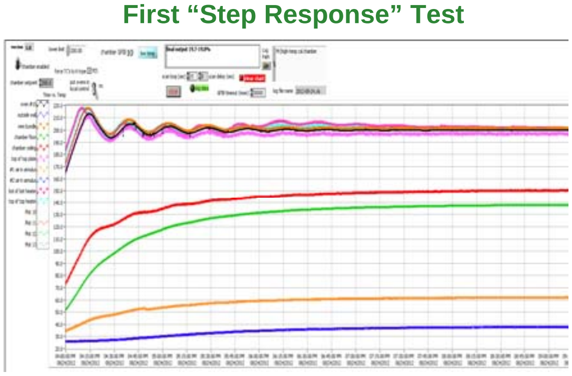
- RC time constant of outer walls of chamber

$$\begin{aligned}\tau &= R_{conv}C_{th}\\ &= 0.23\cdot 15100\\ &= 3400\ s\end{aligned}$$

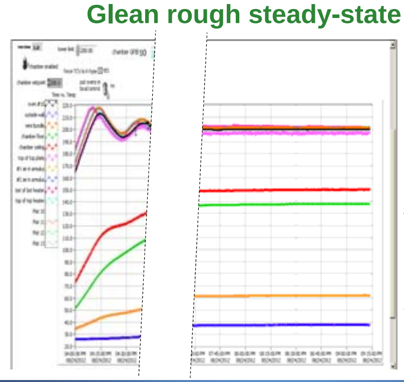
Commercial PID Controller



First "Step Response" Test



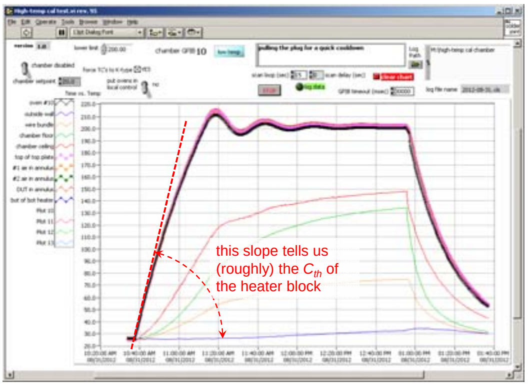
Glean rough steady-state values

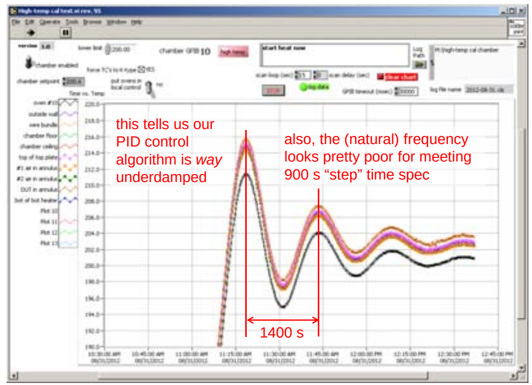


power	230	W
output %	20%	
heater	200	°C
inside wall	144	°C
outside wall	39	°C
ambient	25	°C

Thermal Resistances

heater-inside	1.22	°C/W
inside-outside	2.28	°C/W
outside	0.30	°C/W
total	3.80	°C/W

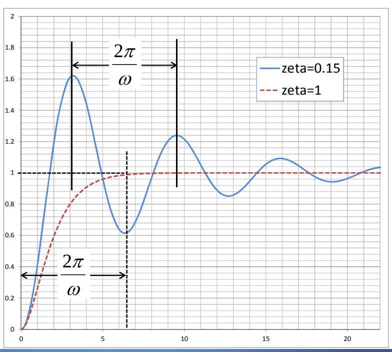




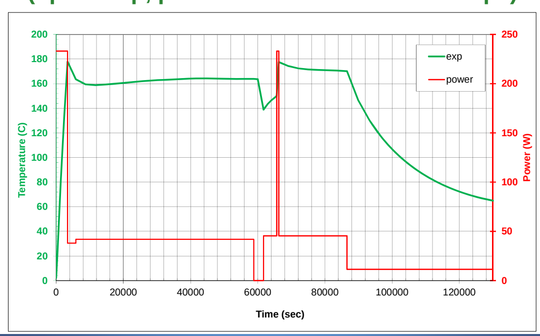
PID controller response

damped frequency within 1% of natural frequency

implies even critically damped system won't settle any faster than about $2\pi/\omega$

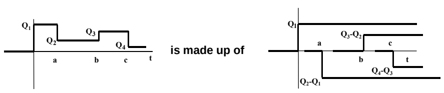


Trial Run of Temperature Chamber
(open loop, power constant in various steps)

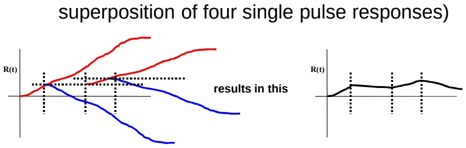


Recall How to Handle Arbitrary Pulses

Four power steps decomposed into infinite steps



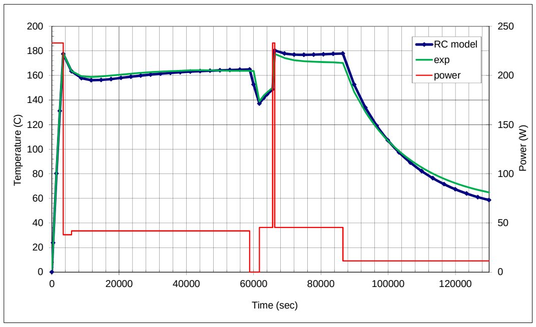
Temperature response constructed from superposition of four single pulse responses)



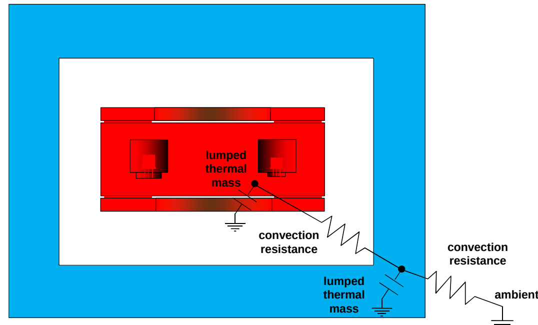
Excel Mechanics of RC Fit

1. A Foster RC ladder network can be computed easily in Excel with a single formula (and appropriate inputs)
2. Choose a set number of points/decade (equal spacing on log scale)
3. Compute error at each time between model and measured curve
4. Use Excel's "Solver" to minimum the root-mean-squared error over the entire curve
5. Choose constraints wisely

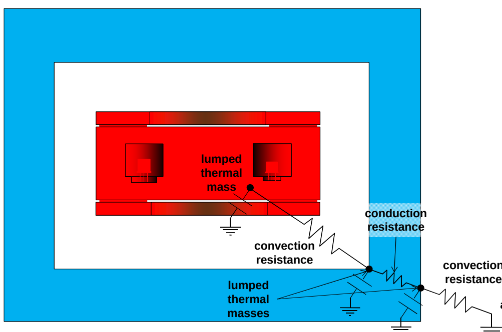
Fitting RC Model to Step Experiment Data



Very Simple (2-node) Thermal RC Model



Simple (3-node) Thermal RC Model



RC Models Fit to Data

2-rung fit (RMSE = 0.000208)

Foster			Cauer	
Tau	**Amp**	**C's**	**R's**	**C's**
3850	0.31	12300	2.33	3740
19600	3.65	5380	1.63	5320

3-rung fit (RMSE = 0.000250)

Foster			Cauer	
Tau	**Amp**	**C's**	**R's**	**C's**
1880	0.002	855000	2.32	3740
3880	0.31	12400	1.47	5100
19600	3.65	5390	0.17	12800

Consider the simple RC products:

2-rung: 8730, 8650

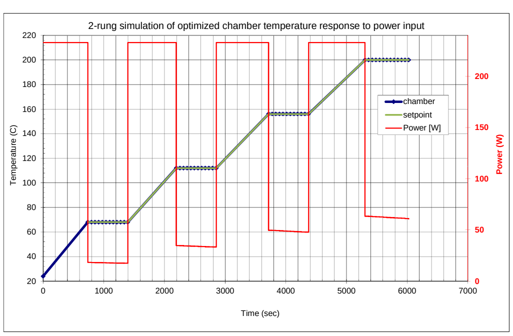
3-rung: 8700, 7500, 2200

C_{th}-inner heater assy = 3270 J/C

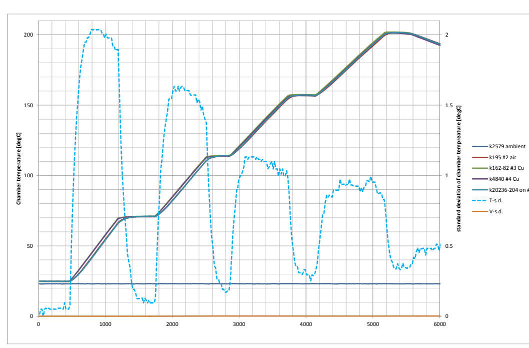
C_{th}-inside walls of outer shell = 4400 J/C

C_{th}-outside wall of outer shell = 10700 J/C

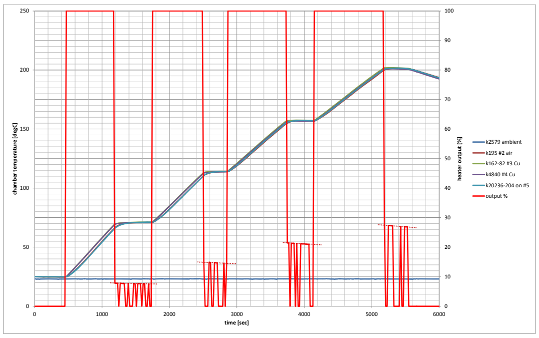
2-Rung RC Simulation of Optimized Response



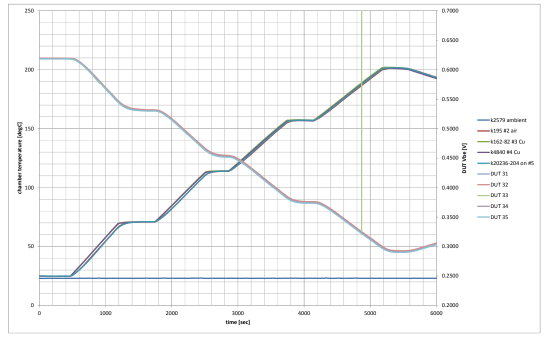
Actual Open-Loop Chamber Response



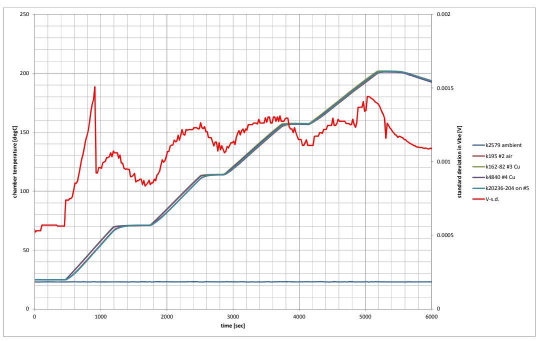
Chamber Response / Power Input



DUT Temperature Profile vs. Average V_f



V_f st.dev. vs. Chamber Temperature Profile



BONUS

If you're interested, and willing to spend another 10-15 minutes …

How to harness this math in Excel®

Array Formulas and Tables

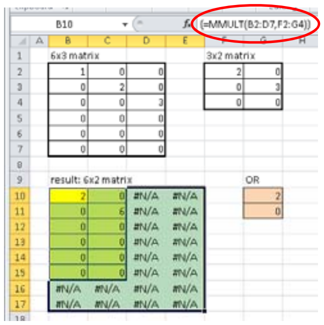
- Array formulas tell Excel
  - apply an operation over a range of cells
  - produce output over (potentially) a range of cells
  - (warning: array formulas applied to AND / OR may surprise you)
- Tables
  - uses a single formula but applies it to a range of values
- With an array formula, you can compute the response of an entire RC ladder in a single formula (for a given time)
- With a carefully designed layout, you can track the response of a train of pulses, one cell for each pulse
- The SUM function can give you the cumulative response of the entire ladder to all pulses (at a given time)
- With a TABLE, you can plot the time history of the whole thing

Some Useful Array and Matrix Functions

- SUM(arguments,…)
  - sums all cells of arguments into single cell result
- MMULT(arg1(m x n) , arg2(n x q))
  - computes matrix multiplication of arg1 by arg2; number of columns of arg1 must number of rows of arg1; result is (m x q) matrix
  - if not entered as array formula, only (1,1) result is returned
- MINVERSE(argument)
  - computes inverse of argument, which must be square matrix
- TRANSPOSE(argument)
  - returns (n x m) matrix for (m x n) argument
- SUMSQ(arguments,…)
  - computes sum of squares of arguments into single cell result

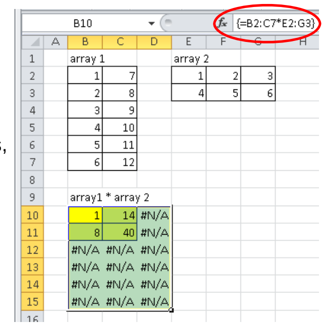
Array Formulas and Matrix Functions

- Array formulas
  - Ctrl-Shift-Enter to create
  - { braces appear around the formula after entered }
- For example, MMULT
  - yellow area is one formula, but larger than needed
  - beige area is same formula, but only big enough for subset of result



Multiplying Blocks of Cells

- If you just "multiply" two areas (arrays of cells), the size of the result is determined by the intersection of the inputs, and the individual cells are simply the products of the corresponding input values, cell by cell. It's nothing like MMULT!



Multiple independent variables

junction temperature vector

theta matrix assembled from simplified subsystems

power input vector

$$\begin{Bmatrix}T_{j1}\\ \vdots \\ T_{jm}\end{Bmatrix} = \begin{bmatrix}\theta_{J1A} & \Psi_{12} & \cdots & \Psi_{1n}\\ \Psi_{12} & & & \Psi_{2n}\\ \vdots & & & \vdots \\ \Psi_{1n} & \Psi_{2n} & \cdots & \theta_{JnA}\end{bmatrix}\begin{Bmatrix}q_1\\ \vdots \\ q_n\end{Bmatrix} + \overline{T}_a$$

$$\overline{\mathbf{T}}_J = [\boldsymbol{\theta}_{JA}]\{\mathbf{q}\} + \overline{\mathbf{T}}_a$$

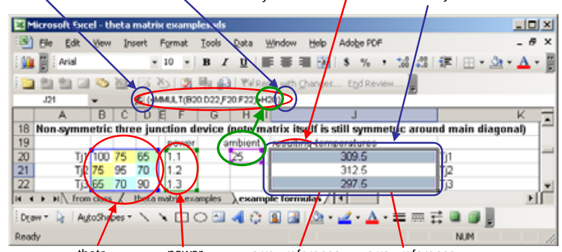
3x3 theta matrix, 3x1 power vector Excel® math

{=array formula notation}

obtained by using *Ctrl-Shift-Enter* rather than ordinary *Enter*

Matrix **MULT**iply

multi-cell placement of array formula



theta matrix

power vector

array reference to theta matrix

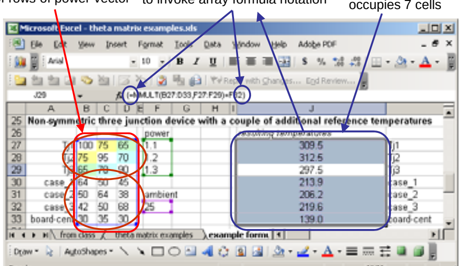
array reference to power vector

7x3 theta matrix, 3x1 power vector Excel® math

theta matrix is no longer square – # of columns still must equal # of rows of power vector

don't forget to use *Ctrl-Shift-Enter* to invoke array formula notation

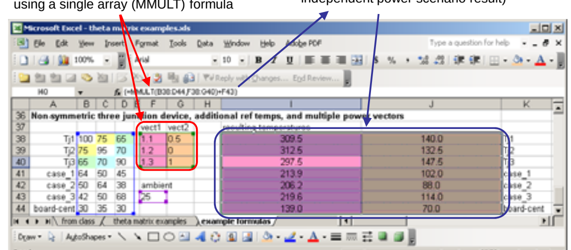
array formula now occupies 7 cells



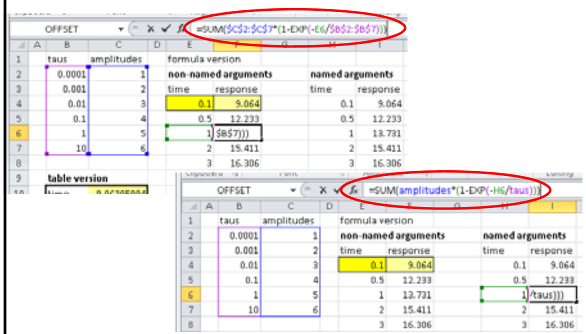
7x3 theta matrix, 3x2 power vector Excel® math

power "vector" is now a 3x**2** array – each column is a different power scenario, yet both are still processed using a single array (MMULT) formula

the single MMULT array formula now occupies 7 rows and **2** columns (one column for each independent power scenario result)

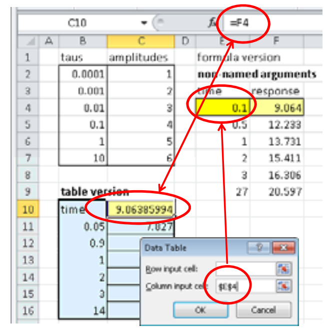


Using Unnamed and Named Input Values



Using Table versus Individual Formulas

- To create table, select entire blue region
- Note light yellow cell can be a formula of its own, or a reference to a formula somewhere else (e.g. cell F4)
- In this example, the "column input cell" is cell E4, because that's the variable argument for the formula in F4

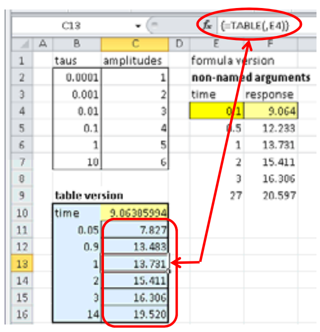


Using Table versus Individual Formulas

- Once table has been created, note that selecting any individual cell below the column formula give you the same

  {=TABLE(,E4)}

  array formula



Putting it Together for a Sequence of Pulses

power data

Power steps Time (sec)	D1
0	0.5
0.01	0
0.02	0.5
0.025	0.25
0.04	0
0.05	0
0.06	0
0.07	0
0.08	0
0.09	0
0.1	0
0.101	0.1
0.102	0.25
0.103	0.5
0.104	0.25
0.105	0.1
0.106	0
0.17	0
0.18	0
0.19	0.5

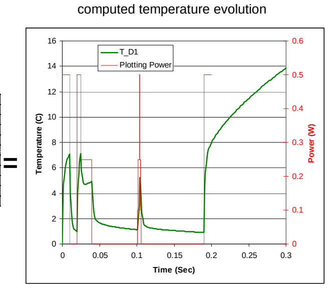
\+

Foster model

	R	Tau
1	0.076	1.00E-06
2	0.164	1.00E-05
3	0.518	1.00E-04
4	0.243	1.00E-03
5	10	1.45E-03
6	0.385	1.98E-03
7	2	0.01
8	10	0.1
9	50	1
10	100	10

=

computed temperature evolution



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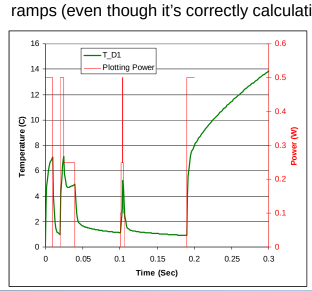
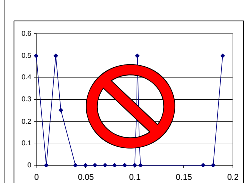
- With a carefully designed layout, you can track the response of a train of pulses, one cell for each pulse
- The SUM function can give you the cumulative response of the entire ladder to all pulses (at a given time)
- With a table, you can plot the time history of the whole thing
- compute error at selected time points between model and measured curve
- use Excel's "Solver" to minimum the root-mean-squared error over the entire curve

A Note on Graphing the Power Steps

If you want the plot to *look* like steps, you have to create a special version of the input data table, otherwise you get what looks like ramps (even though it's correctly calculation step responses)





Orders of Magnitude and Rungs

- It is a rare transient curve that cannot be followed very accurately with time constants no closer than about 1 order of magnitude apart. This means that you need only about one rung per decade of transient response.

Same Method Can be Applied to Systems with Multiple Heat Sources

- Each heat source has its own Foster description
  - of its own self heating
  - of its interaction with every other location of interest
  - different Foster descriptions don't have to share common time constants
  - it's OK (in fact, it's virtually required) for interaction Foster descriptions to have some negative amplitudes
- The "master" power input table has to introduce a new "step" whenever *any* power input changes, whether you're computing the temperature of that source or simply the effect of that (and every) source in the system; unchanging sources simply have a step of zero

References

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8. ON Semiconductor App Note AND8219/D "*How to Generate Square Wave, Constant Duty Cycle, Transient Response Curves*," Roger Stout, 2006
9. ON Semiconductor App Note AND8220/D "*How To Use Thermal Data Found in Data Sheets*," Roger Stout, 2006